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Approximations to generalized renewal measures☆

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Abstract

Let $\{Z_j, j \geqslant 1\}$ be a sequence of nonnegative continuous random variables. Given an arbitrary function $g:[0,\infty) \to [0,\infty)$, a renewal function associated with this sequence is defined as

$$S(b) = \sum_{j=1}^{\infty} g(j)P\{Z_j < b\}, \qquad b > 0.$$

Due to possible complexity of calculating the probabilities $P\{Z_j < b\}$, computation of $S(b)$ is often intractable.

Consider a sequence of positive numbers $\{m_j, j \geqslant 1\}$ and define

$$S^*(b) = \sum_{j=1}^{\infty} g(j)I\{m_j < b\}.$$

Clearly, $S^*(b)$ is much easier to calculate than $S(b)$. We propose $S^*(b)$ as an approximation to $S(b)$, and present a bound on the difference between them. Under certain circumstances, our finding is an improvement of a result of Alsmeyer, both in sharpness of the bound and in extension to more general sequences $\{Z_j\}$. The methods employed are Tauberian in nature.
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1. Introduction

Let $\{Z_j,\ j \geqslant 1\}$ be a sequence of nonnegative continuous random variables. Given an arbitrary function $g:[0,\infty) \to [0,\infty)$, a generalized renewal function associated with this sequence is defined as

$$S(b) = \sum_{j=1}^{\infty} g(j)P\{Z_j < b\}, \quad b > 0. \tag{1.1}$$

$S(b)$ can be regarded as a generalized renewal measure of $\{Z_j,\ j \geqslant 1\}$ (which for $g \equiv 1$ reduces to the ordinary renewal measure). Alsmeyer (1992) studied $S(b)$ for the special case that the sequence $\{Z_j\}$ is a sequence of partial sums of iid random variables with nonnegative mean μ. Alsmeyer's Theorem 1 states that under certain conditions on the underlying iid variables and under the condition that $g(t)$ is regular varying at infinity with exponent $\alpha > -1$,

$$S(b) \sim \frac{bg(b)}{(\alpha+1)\mu^{\alpha+1}} \quad \text{as } b \to \infty.$$

Omey and Teugels (2002) also studied weighted renewal functions (of the form (1.1)) under the conditions of Alsmeyer, with the added provision that the iid variables whose partial sums are $\{Z_l\}$ be positive random variables. In addition to elementary-type renewal theorems, Alsmeyer as well as Omey and Teugels have Blackwell-type results.

In this note, we regard a much more general stochastic sequence $\{Z_j,\ j \geqslant 1\}$. We choose a sequence of constants $\{m_j,\ j \geqslant 1\}$ and define

$$S^*(b) = \sum_{j=1}^{\infty} g(j)I\{m_j < b\}, \quad b > 0 \tag{1.2}$$

(where $I\{\cdot\}$ is the indicator function). In applications, the usual choice will be $m_j = EZ_j$. We define a function $U(b)$ that in many cases is bounded, and prove that the difference $S(b) - S^*(b)$ is bounded by a multiple of $1 + U(b)$. Our methods are based on a Tauberian approach and differ completely from those in Alsmeyer (1992).

2. Main result

Let $\{Z_j,\ j \geqslant 1\}$ be a sequence of nonnegative random variables whose distributions are continuous. Given an arbitrary function $g:[0,\infty) \to [0,\infty)$, a renewal function associated with this sequence is defined as

$$S(b) = \sum_{j=1}^{\infty} g(j)P\{Z_j < b\}, \quad b > 0. \tag{2.1}$$

We are interested in deriving an approximation to $S(b)$ as $b \to \infty$ having a bounded remainder term. Consider a sequence of positive numbers $\{m_j,\ j \geqslant 1\}$

and define

$$S^*(b) = \sum_{j=1}^{\infty} g(j) I\{m_j < b\}. \tag{2.2}$$

Theorem 1 presents a bound on the difference between $S(b)$ and $S^*(b)$.

Theorem 1. *Define*

$$U(b) = \frac{1}{b^2} \sum_{j=1}^{\infty} \mathrm{e}^{-m_j/b} g(j) E(Z_j - m_j)^2$$

and assume

$$U(b) < \infty$$

and

$$\sum_{j=1}^{\infty} \mathrm{e}^{-m_j/b} g(j) < \infty \quad \text{for all } b \in (0, \infty), \tag{2.3}$$

$$\lim_{b\to\infty} \lim_{T\to\infty} \sup \frac{\sum_{j=1}^{\infty} g(j) \mathrm{e}^{-m_j/b} \min(1, (T(b - m_j))^{-2})}{1 + U(b)} < \infty, \tag{2.4}$$

$$\sum_{j=1}^{\infty} g(j) P\{|Z_j - m_j| > \delta m_j\} < \infty \quad \text{for some } \delta > 0. \tag{2.5}$$

Then, as $b \to \infty$,

$$S(b) = S^*(b) + \mathrm{O}(1 + U(b)).$$

Remark 1. Note that the result of Theorem 1 remains valid if the condition $Z_j \geqslant 0$ is relaxed, as long as assumptions (2.3)–(2.5) are satisfied with Z_j replaced by $Z_j^+ = \max(0, Z_j)$ (since $Z_j \leqslant b$ iff $Z_j^+ \leqslant b$ for $b > 0$).

Remark 2. The reason why we write $\mathrm{O}(1 + U(b))$ in Theorem 1 is that $U(b)$ may be $\mathrm{o}(1)$ as $b \to \infty$.

Remark 3. As mentioned above, Alsmeyer (1992) studied $S(b)$ when $\{x_i\}$ is an iid sequence with nonnegative mean μ and $Z_j = \sum_{i=1}^{j} x_i$. Alsmeyer required that g is a regular varying function. Our Theorem 1 requires a strong condition on the moments of x_i (at least Ex_i^2 must be finite, whereas Alsmeyer has $E(x_1^-)^2 g(x_1^-) < \infty$), but relaxes somewhat the condition on g. For example, if $g(j) = j^{\log(j)}$ (which is not regular varying) and x_1 has a moment generating function, the conditions of our Theorem 1 are satisfied when $m_j = \mu j$. In this case, (2.5) is easily seen to be satisfied by using standard considerations of large deviations; (2.3) holds trivially and for fixed $b>0$

$$\begin{aligned}
&\lim_{T\to\infty}\sup \frac{\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-m_j/b}\min(1,(T(b-m_j))^{-2})}{1+U(b)}\\
&\quad\leqslant \frac{(b/\mu)^{\log(b/\mu)}\mathrm{e}^{-1}}{1+U(b)} \leqslant \frac{(b/\mu)^{\log(b/\mu)}\mathrm{e}^{-1}}{\frac{\mathrm{Var}(x_1)}{b^2}\sum_{j>b/\mu} j^{\log(j)}\mathrm{e}^{-\mu j/b}\, j}\\
&\quad\leqslant \frac{b^2\mathrm{e}^{-1}}{\mathrm{Var}(x_1)\sum_{j>b/\mu}\mathrm{e}^{-\mu j/b}\, j}\\
&\quad\leqslant \frac{\mu}{\mathrm{Var}(x_1)(b(\mathrm{e}^{\mu/b}-1)+\mu)}\, b^2(1-\mathrm{e}^{-\mu/b})^2 \to_{b\to\infty} \frac{\mu^2}{2\,\mathrm{Var}(x_1)}
\end{aligned}$$

which accounts for (2.5).

3. Sketch of proof

The assertion of Theorem 1 is a consequence of Lemmas 4.2 and 4.3 stated in Section 4. These lemmas relate the asymptotic behavior of $S(b)$ and $S^*(b)$ as $b\to\infty$ to that of the functions

$$\varphi(z)=\sum_{j=1}^{\infty} g(j)E\mathrm{e}^{-zZ_j}, \qquad \varphi^*(z)=\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-zm_j}, \tag{3.1}$$

when the real part of the complex argument z approaches 0. It is established in Lemma 4.2 that, for large b, $S(b)-S^*(b)$ can be approximated by $\lim_{T\to\infty}(\widetilde{S}(T,b)-\widetilde{S}^*(T,b))$, where

$$\widetilde{S}(T,b)=\int_{-T}^{T}\psi(b^{-1},b,\mathrm{i}t)\varphi(b^{-1}+\mathrm{i}t)\,\mathrm{d}t,$$

$$\widetilde{S}^*(T,b)=\int_{-T}^{T}\psi(b^{-1},b,\mathrm{i}t)\varphi^*(b^{-1}+\mathrm{i}t)\,\mathrm{d}t$$

and

$$\psi(s,v,z)=\frac{1}{2\pi(T^2+s^2)^2}\,\frac{(T^2+z^2)^2}{s+z}\,\mathrm{e}^{(s+z)v} \tag{3.2}$$

is a weight function, T is some positive constant. The error of this approximation is bounded by $\mathrm{O}(1+U(b))$ as $b\to\infty$. The proof of Lemma 4.2 depends on a representation of the integral

$$K(v)=\int_{-T}^{T}\psi(s,v,\mathrm{i}t)\,\mathrm{d}t.$$

Considerations similar to Lemma 4.1 have been employed in other contexts (see, for example, Lemma 2.1.1 in Subkhankulov, 1976). It is then shown in Lemma 4.3 that applying a Taylor series expansion and making use of the assumptions of Theorem 1 yields the boundedness of the difference $\lim_{T\to\infty}(\widetilde{S}(T,b)-\widetilde{S}^*(T,b))$, as $b\to\infty$. In view of Lemma 4.2, this implies that $S(b)-S^*(b)$ is bounded by $\mathrm{O}(1+U(b))$. This completes the proof of Theorem 1.

The method used in the proof of Theorem 1 can be formally related to Tauberian theorems for the Laplace–Stieltjes transform. In fact, it is straightforward to see that $\varphi(z)$ and $\varphi^*(z)$ are the Laplace–Stieltjes transforms of the renewal functions $S(b)$ and $S^*(b)$, respectively. However, instead of formulating a Tauberian theorem and then verifying its conditions, we adapt a direct approach based on introducing "smooth" approximations to $S(b)$ and $S^*(b)$ (namely, $\widetilde{S}(T,b)$ and $\widetilde{S}^*(T,b)$, as $T\to\infty$) and then making use of a Taylor expansion to extract a bounded remainder. This direct approach allows us to achieve the goal of establishing the boundedness of $S(b)-S^*(b)$ under weak assumptions.

4. Details of proof

Lemma 4.1. *Let $s=1/b<T$ for some $T>0$ and, for any real v, define*

$$K(v)=\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s+\mathrm{i}t}\,\mathrm{e}^{(s+\mathrm{i}t)v}\,\mathrm{d}t, \tag{4.1}$$

where $\mathrm{i}=\sqrt{-1}$. Then

$$K(v)=\begin{cases}1+\mathrm{O}(\mathrm{e}^{sv}\min(1,(Tv)^{-2})) & \text{if } v\geqslant 0,\\ \mathrm{O}(\mathrm{e}^{sv}\min(1,(Tv)^{-2})) & \text{if } v<0,\end{cases} \tag{4.2}$$

where the bound in $\mathrm{O}(\cdot)$ does not involve T, v and s.

Proof. Assume first that $v\geqslant 0$ and consider the integral

$$\frac{1}{\mathrm{i}}\int_A \psi(s,v,z)\,\mathrm{d}z, \tag{4.3}$$

where A is some contour (for example, $A=\{\zeta\colon\ \zeta+s=\sqrt{T^2+s^2}\mathrm{e}^{\mathrm{i}\vartheta},\ \mathrm{Re}\,\zeta<0\}\cup\{\zeta\colon\ \mathrm{Re}\,\zeta=0,\ -T\leqslant\mathrm{Im}\,\zeta\leqslant T\}$) and $\psi(s,v,z)$ is defined in (3.2). It is easy to see that $z=-s$ is the only singularity of the integrand $\psi(s,v,z)$ in (4.3) as a function of z. By the residue theorem of complex analysis,

$$K(v)=2\pi\,\mathrm{res}_{z=-s}\,\psi(s,v,z)-\frac{1}{\mathrm{i}}\int_C \psi(s,v,z)\,\mathrm{d}z, \tag{4.4}$$

where $\mathrm{res}_{z=-s}\,\psi(s,v,z)$ denotes the residue of $\psi(s,v,z)$ at $z=-s$ and the contour C is defined as $C=\{\zeta\colon\ \zeta+s=r\mathrm{e}^{\mathrm{i}\vartheta},\ \mathrm{Re}\,\zeta<0\}$ with $r=\sqrt{T^2+s^2}$. In other words, C is the part of the circle $\zeta+s=r\mathrm{e}^{\mathrm{i}\vartheta}$, $0\leqslant\vartheta\leqslant 2\pi$, to the left of the imaginary axis. Since $z=-s$ is a simple pole of $\psi(s,v,z)$, one can show that $\mathrm{res}_{z=-s}\,\psi(s,v,z)=1/(2\pi)$. Regarding the second term on the right-hand side of (4.4), note that, along the contour C, $|s+z|=r$, $|z|\leqslant 2T$ and $|\mathrm{e}^{zv}|\leqslant 1$ since $v\geqslant 0$ and $\mathrm{Re}\,z<0$.

This implies that

$$\begin{aligned}
&\frac{1}{\mathrm{i}}\int_C \psi(s,v,z)\,\mathrm{d}z\\
&=\frac{\mathrm{e}^{sv}}{2\pi r^4}\int_{\theta\in l_C}(r^2+r^2\mathrm{e}^{2\theta\mathrm{i}}-2sr\mathrm{e}^{\mathrm{i}\theta})^2\mathrm{e}^{(-s+r(\cos(\theta)+\mathrm{i}\sin(\theta)))v}\,\mathrm{d}\theta\\
&=\frac{\mathrm{e}^{sv}}{2\pi}\int_{\theta\in l_C}\left(1+\mathrm{e}^{2\theta\mathrm{i}}-2\,\frac{s}{r}\,\mathrm{e}^{\mathrm{i}\theta}\right)^2\mathrm{e}^{(r\cos(\theta)-s)v}\mathrm{e}^{\mathrm{i}r\sin(\theta)v}\,\mathrm{d}\theta\\
&=\mathrm{O}\left(\frac{\mathrm{e}^{sv}}{2\pi}\int_{\theta\in l_C}\left|\left(1+\mathrm{e}^{2\theta\mathrm{i}}-2\,\frac{s}{r}\,\mathrm{e}^{\mathrm{i}\theta}\right)^2\right|\mathrm{d}\theta\right)=\mathrm{O}\left(\frac{8\mathrm{e}^{sv}}{\pi}\int_{\theta\in l_C}\mathrm{d}\theta\right)\\
&=\mathrm{O}(\mathrm{e}^{sv}),
\end{aligned}\tag{4.5}$$

where

$$l_C=\left\{\theta\in(\theta_0,\theta_1),\ 0\leqslant\theta_0,\theta_1\leqslant 2\pi,\ r\cos(\theta)-s<0,\ \cos(\theta_0)=\cos(\theta_1)=\frac{s}{r},\right.$$
$$\left.\sin(\theta_0)=\frac{T}{r},\ \sin(\theta_1)=-\frac{T}{r}\right\}.$$

On the other hand, integrating the second term on the right-hand side of (4.4) by parts two times and noting that $\psi(s,v,\mathrm{i}T)=\psi(s,v,-\mathrm{i}T)=0$, we have

$$\begin{aligned}
\frac{1}{\mathrm{i}}\int_C \psi(s,v,z)\,\mathrm{d}z&=\frac{\mathrm{e}^{sv}}{\mathrm{i}v2\pi(T^2+s^2)^2}\int_C\frac{(T^2+z^2)^2}{s+z}\,\mathrm{d}\mathrm{e}^{zv}\\
&=\frac{\mathrm{e}^{sv}}{\mathrm{i}v2\pi(T^2+s^2)^2}\left(\frac{(T^2+z_0^2)^2}{s+z_0}\,\mathrm{e}^{z_0v}\Big|_{z_0=T\mathrm{i}}^{z_0=-T\mathrm{i}}\right.\\
&\quad\left.-\int_C\mathrm{e}^{zv}\,\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(T^2+z^2)^2}{s+z}\right)\mathrm{d}z\right)\\
&=\frac{-\mathrm{e}^{sv}}{\mathrm{i}v^2 2\pi(T^2+s^2)^2}\left(\int_C\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(T^2+z^2)^2}{s+z}\right)\mathrm{d}\mathrm{e}^{zv}\right)\\
&=\frac{-\mathrm{e}^{sv}}{\mathrm{i}v^2 2\pi(T^2+s^2)^2}\left(\mathrm{e}^{z_0v}\,\frac{\mathrm{d}}{\mathrm{d}z_0}\left(\frac{(T^2+z_0^2)^2}{s+z_0}\right)\Big|_{z_0=T\mathrm{i}}^{z_0=-T\mathrm{i}}\right.\\
&\quad\left.-\int_C\mathrm{e}^{zv}\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{(T^2+z^2)^2}{s+z}\right)\mathrm{d}z\right)\\
&=\mathrm{O}\left(\frac{\mathrm{e}^{sv}}{(T^2v)^2}\int_C\mathrm{e}^{zv}\,\frac{d^2}{dz^2}\left(\frac{(T^2+z^2)^2}{s+z}\right)\mathrm{d}z\right)
\end{aligned}$$

$$= \mathrm{O}\left(\frac{\mathrm{e}^{sv}}{(T^2v)^2}\int_C\left[\frac{T^2}{r}+\frac{|z|^2}{r}+\frac{|z|^3}{r^2}+\frac{|z|^4}{r^3}\right]\mathrm{d}l\right)$$

$$= \mathrm{O}((Tv)^{-2}\mathrm{e}^{sv}). \tag{4.6}$$

From (4.4)–(4.6), one can infer that $K(v) = 1 + \mathrm{O}(\mathrm{e}^{sv}\min(1,(Tv)^{-2}))$ if $v \geqslant 0$.

In case $v < 0$, the assertion of the lemma follows from integrating $\psi(s,v,z)$ with respect to z along the contour $D = \{\zeta\colon\ \zeta + s = r\mathrm{e}^{\mathrm{i}\vartheta},\ \mathrm{Re}\,\zeta > 0\}$ and noting that, by Cauchy's fundamental theorem, $K(v) = -(1/\mathrm{i})\int_D \psi(s,v,z)\,\mathrm{d}z$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $s = 1/b < T$ for some $T > 0$. Then, under the assumptions of Theorem 1*

$$\overline{\lim_{b\to\infty}}\ \overline{\lim_{T\to\infty}}\left|\frac{\int_{-T}^{T}\psi(s,b,\mathrm{i}t)(\varphi(s+\mathrm{i}t)-\varphi^*(s+\mathrm{i}t))\,\mathrm{d}t-(S(b)-S^*(b))}{1+U(b)}\right| < \infty. \tag{4.7}$$

Proof. Note that $P\{Z_j < b\} = P\{\mathrm{e}^{-Z_j/b} > \mathrm{e}^{-1}\}$, $j \geqslant 1$, and apply Markov's inequality to obtain from Conditions (2.3) and (2.5) that $S(b) \leqslant \mathrm{e}\sum_{j=1}^{\infty} g(j)E\mathrm{e}^{-Z_j/b} < \infty$ and, uniformly in $t > 0$, $E|\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-(s+\mathrm{i}t)Z_j}| < \infty$. Therefore, interchanging integration and expectation we obtain that

$$\int_{-T}^{T}\psi(s,b,\mathrm{i}t)\varphi(s+\mathrm{i}t)\,\mathrm{d}t = \int_{-T}^{T}\psi(s,b,\mathrm{i}t)E\left(\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-(s+\mathrm{i}t)Z_j}\right)\mathrm{d}t$$

$$= E\left(\sum_{j=1}^{\infty} g(j)\int_{-T}^{T}\psi(s,b-Z_j,\mathrm{i}t)\,\mathrm{d}t\right)$$

$$= R_1 + R_2, \tag{4.8}$$

where

$$R_1 = E\sum_{j=1}^{\infty} g(j)K(b-Z_j)I\{Z_j \leqslant b\},$$

$$R_2 = E\sum_{j=1}^{\infty} g(j)K(b-Z_j)I\{Z_j > b\},$$

$K(v)$ is defined by (4.1) in Lemma 4.1. Let $s = 1/b$ and $v = b - Z_j$. Applying (4.2) in Lemma 4.1 with $v \geqslant 0$ for R_1 and $v < 0$ for R_2 we have

$$R_1 = S(b) + \mathrm{O}\left(\sum_{j=1}^{\infty} g(j)E\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2})I\{Z_j \leqslant b\}\right), \tag{4.9}$$

$$R_2 = \mathrm{O}\left(\sum_{j=1}^{\infty} g(j)E\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2})I\{Z_j > b\}\right) \tag{4.10}$$

since $\mathrm{e}^{(b-Z_j)/b} = \mathrm{e}^{1-Z_j/b}$.

If $0 < \delta < 1$ then by Condition (2.5) of Theorem 1 we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} g(j)E\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2}) \\
&\quad\leqslant \sum_{j=1}^{\infty} g(j)\mathrm{e}^{-s\delta m_j}\,E\min(1,(T(b-Z_j))^{-2}) \\
&\qquad + \sum_{j=1}^{\infty} g(j)E\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2})I\{Z_j < \delta m_j\} \\
&\quad\leqslant \sum_{j=1}^{\infty} g(j)\mathrm{e}^{-s\delta m_j}\,E\min(1,(T(b-Z_j))^{-2}) + \mathrm{O}(1),
\end{aligned} \tag{4.11}$$

where $\mathrm{O}(1)$ is independent of b, T. By virtue of (2.3) for fixed $b > 0$, $\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-s\delta m_j} < \infty$. Therefore, for fixed $b > 0$,

$$\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-s\delta m_j}\,E\min(1,(T(b-Z_j))^{-2}) \to_{T\to\infty} 0$$

so that

$$\lim_{b\to\infty}\lim_{T\to\infty}\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-s\delta m_j}\,E\min(1,(T(b-Z_j))^{-2}) = 0. \tag{4.12}$$

Combining (4.11) and (4.12), we conclude that

$$\overline{\lim_{b\to\infty}}\;\overline{\lim_{T\to\infty}}\sum_{j=1}^{\infty} g(j)E\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2}) < \infty. \tag{4.13}$$

Applying (4.13) to (4.9) and (4.10), we have

$$\begin{aligned}
&\overline{\lim_{b\to\infty}}\;\overline{\lim_{T\to\infty}}\quad |R_1 - S(b)| < \infty, \\
&\overline{\lim_{b\to\infty}}\;\overline{\lim_{T\to\infty}}\quad |R_2| < \infty.
\end{aligned} \tag{4.14}$$

It now follows from (4.8) that

$$\overline{\lim_{b\to\infty}}\;\overline{\lim_{T\to\infty}}\quad \left|\int_{-T}^{T} \psi(s,b,\mathrm{i}t)\varphi(s+\mathrm{i}t)\,\mathrm{d}t - S(b)\right| < \infty. \tag{4.15}$$

One can show in a similar manner from Lemma 4.1 that

$$\int_{-T}^{T} \psi(s,b,\mathrm{i}t)\varphi^*(s+\mathrm{i}t)\,\mathrm{d}t = S^*(b) + \mathrm{O}\left(\sum_{j=1}^{\infty} g(j)\mathrm{e}^{-sm_j}\min(1,(T(b-m_j))^{-2})\right),$$

so that by condition (2.4)

$$\overline{\lim_{b\to\infty}}\ \overline{\lim_{T\to\infty}}\ \left|\frac{\int_{-T}^{T} \psi(s,b,\mathrm{i}t)\varphi^*(s+\mathrm{i}t)\,\mathrm{d}t - S^*(b)}{1+U(b)}\right| < \infty.$$

It is clear that this equality and (4.15) imply (4.7). The proof of Lemma 4.2 is now complete. □

Lemma 4.3. *Let $s = 1/b < T$. Then, under the assumptions of Theorem* 1

$$\overline{\lim_{b\to\infty}}\ \overline{\lim_{T\to\infty}}\left|\frac{\int_{-T}^{T} \psi(s,b,\mathrm{i}t)(\varphi(s+\mathrm{i}t)-\varphi^*(s+\mathrm{i}t))\,\mathrm{d}t}{1+U(b)}\right| < \infty. \tag{4.16}$$

Proof. By Lemma 4.1

$$\begin{aligned}
&\sum_{j=1}^{\infty} g(j)\,\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s+\mathrm{i}t}(\mathrm{e}^{(s+\mathrm{i}t)(b-Z_j)}-\mathrm{e}^{(s+\mathrm{i}t)(b-m_j)})\,\mathrm{d}t\\
&\quad=\sum_{j=1}^{\infty} g(j)(I\{Z_j<b\}-I\{m_j<b\}\\
&\qquad+\mathrm{O}(\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2}))\\
&\qquad+\mathrm{O}(\mathrm{e}^{-sm_j}\min(1,(T(b-m_j))^{-2}))).
\end{aligned} \tag{4.17}$$

On the other hand,

$$\begin{aligned}
&\sum_{j=1}^{\infty} g(j)\,\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s+\mathrm{i}t}\,\mathrm{e}^{(s+\mathrm{i}t)(b-Z_j)}\,\mathrm{d}t\\
&\quad=\sum_{j=1}^{\infty} g(j)\,\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2(s-\mathrm{i}t)}{(s+\mathrm{i}t)(s-\mathrm{i}t)}\,\mathrm{e}^{(s+\mathrm{i}t)(b-Z_j)}\,\mathrm{d}t\\
&\quad=\sum_{j=1}^{\infty} g(j)\,\frac{1}{2\pi(T^2+s^2)^2}\left(\int_{-T}^{T}\frac{(T^2-t^2)^2}{s^2+t^2}\,\mathrm{e}^{s(b-Z_j)}(s\cos(t(b-Z_j))\right.\\
&\qquad+t\sin(t(b-Z_j)))\,\mathrm{d}t\\
&\qquad\left.+\mathrm{i}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s^2+t^2}\,\mathrm{e}^{s(b-Z_j)}(s\sin(t(b-Z_j))-t\cos(t(b-Z_j)))\,\mathrm{d}t\right).
\end{aligned} \tag{4.18}$$

Now, by the anti-symmetry of the integral

$$\int_{-T}^{T} \frac{(T^2-t^2)^2}{s^2+t^2}\, \mathrm{e}^{s(b-Z_j)}(s\sin(t(b-Z_j)) - t\cos(t(b-Z_j)))\,\mathrm{d}t = 0,$$

we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} g(j)\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s+\mathrm{i}t}\,\mathrm{e}^{(s+\mathrm{i}t)(b-Z_j)}\,\mathrm{d}t \\
&\quad = \sum_{j=1}^{\infty} g(j)\frac{1}{2\pi(T^2+s^2)^2} \\
&\qquad \times\left(\int_{-T}^{T}\frac{(T^2-t^2)^2}{s^2+t^2}\,\mathrm{e}^{s(b-Z_j)}(s\cos(t(b-Z_j)) + t\sin(t(b-Z_j)))\,\mathrm{d}t\right).
\end{aligned} \tag{4.19}$$

Expanding the function

$$\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s^2+t^2}\,\mathrm{e}^{s(b-u)}(s\cos(t(b-u)) + t\sin(t(b-u)))\,\mathrm{d}t$$

by Taylor series about the point $u = m_j$ yields

$$\begin{aligned}
&\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s^2+t^2}\,\mathrm{e}^{s(b-Z_j)}(s\cos(t(b-Z_j)) + t\sin(t(b-Z_j)))\,\mathrm{d}t \\
&\quad = \frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s^2+t^2}\,\mathrm{e}^{s(b-m_j)}(s\cos(t(b-m_j)) \\
&\qquad + t\sin(t(b-m_j)))\,\mathrm{d}t - \frac{(Z_j-m_j)}{2\pi(T^2+s^2)^2}\int_0^1 \mathrm{e}^{s(b-m_j-y(Z_j-m_j)} \\
&\qquad \times\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\,\mathrm{d}y.
\end{aligned}$$

Combining the result with (4.18) and (4.19), we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} g(j)\,\frac{1}{2\pi(T^2+s^2)^2}\int_{-T}^{T}\frac{(T^2-t^2)^2}{s+\mathrm{i}t}(\mathrm{e}^{(s+\mathrm{i}t)(b-Z_j)} - \mathrm{e}^{(s+\mathrm{i}t)(b-m_j)})\,\mathrm{d}t \\
&\quad = \sum_{j=1}^{\infty} g(j)\,\frac{(m_j-Z_j)}{2\pi(T^2+s^2)^2}\int_0^1 \mathrm{e}^{s(b-m_j-y(Z_j-m_j)} \\
&\qquad \times\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\,\mathrm{d}y.
\end{aligned} \tag{4.20}$$

By definitions (3.1) and (3.2), one can infer from (4.17) and (4.20) that

$$\int_{-T}^{T} \psi(s,b,\mathrm{i}t)(\varphi(s+\mathrm{i}t)-\varphi^*(s+\mathrm{i}t))\,\mathrm{d}t$$

$$=\sum_{j=1}^{\infty} g(j)E\min(I\{Z_j<b\}-I\{m_j<b\}$$

$$+\mathrm{O}(\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2}))$$

$$+\mathrm{O}(\mathrm{e}^{-sm_j}\min(1,(T(b-m_j))^{-2}))$$

$$\times\frac{m_j-Z_j}{2\pi(T^2+s^2)^2}\int_0^1 \mathrm{e}^{s(b-m_j-y(Z_j-m_j))}\int_{-T}^{T}(T^2-t^2)^2$$

$$\times\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\,\mathrm{d}y).$$

Therefore, there exist a constant $w>e$ such than

$$\int_{-T}^{T} \psi(s,b,\mathrm{i}t)(\varphi(s+\mathrm{i}t)-\varphi^*(s+\mathrm{i}t))\,\mathrm{d}t$$

$$\leqslant \sum_{j=1}^{\infty} g(j)E\min\left(w(\mathrm{e}^{-sZ_j}+\mathrm{e}^{-sm_j})\frac{|m_j-Z_j|}{2\pi(T^2+s^2)^2}\left|\int_0^1 \mathrm{e}^{s(b-m_j-y(Z_j-m_j))}\right.\right.$$

$$\left.\left.\times\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\,\mathrm{d}y\right|\right),$$

where the last inequality follows from $I\{\xi<b\}=I\{\mathrm{e}^{-s\xi}>\mathrm{e}^{-1}\}\leqslant \mathrm{e}^{-s\xi+1}$. If $0<\delta<1$ then we have

$$\int_{-T}^{T} \psi(s,b,\mathrm{i}t)(\varphi(s+\mathrm{i}t)-\varphi^*(s+\mathrm{i}t))\,\mathrm{d}t$$

$$\leqslant \sum_{j=1}^{\infty} g(j)E\min\left(w(\mathrm{e}^{-sZ_j}+\mathrm{e}^{-sm_j})\frac{|m_j-Z_j|}{2\pi(T^2+s^2)^2}\left|\int_0^1 \mathrm{e}^{s(b-m_j-y(Z_j-m_j))}\right.\right.$$

$$\left.\left.\times\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\,\mathrm{d}y\right|\right)$$

$$\times(I\{Z_j\geqslant\delta m_j\}+I\{Z_j<\delta m_j\}).$$

Therefore, from condition (2.5) of Theorem 1

$$\int_{-T}^{T} \psi(s,b,\mathrm{i}t)(\varphi(s+\mathrm{i}t)-\varphi^*(s+\mathrm{i}t))\,\mathrm{d}t$$

$$\leqslant \sum_{j=1}^{\infty} g(j)\mathrm{e}^{-\delta s m_j}\, E\min\Bigg(2w, \mathrm{e}\,\frac{|m_j-Z_j|}{2\pi(T^2+s^2)^2}$$

$$\times \int_0^1 \left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|\mathrm{d}y\Bigg)+\mathrm{O}(1). \tag{4.21}$$

Now,

$$E\min\Bigg(2w, \mathrm{e}\,\frac{|m_j-Z_j|}{2\pi(T^2+s^2)^2}$$

$$\times \int_0^1 \left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|\mathrm{d}y\Bigg)$$

$$\leqslant 2wP\{|Z_j-m_j|<\ln(T)^{-1}\}+\ \mathrm{e}E\,\frac{|m_j-Z_j|}{2\pi(T^2+s^2)^2}$$

$$\times \int_0^1 \left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|\,\mathrm{d}y$$

$$\times I\{|Z_j-m_j|\geqslant \ln(T)^{-1}\} \tag{4.22}$$

and

$$\int_0^1 \left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|\,\mathrm{d}y$$

$$=\int_0^1 \left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|$$

$$\times I\{|b-m_j-y(Z_j-m_j)|<T^{-11/20}\}\,\mathrm{d}y$$

$$+\int_0^1 \left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|$$

$$\times I\{|b-m_j-y(Z_j-m_j)|\geqslant T^{-11/20}\}\,\mathrm{d}y,$$

from $I\{|\xi|<a\}\leqslant a^2/\xi^2$, we have

$$\int_0^1 \left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|\mathrm{d}y$$

$$\leqslant 2T^{3.9}\int_0^1 \frac{1}{(b-m_j-y(Z_j-m_j))^2}\,\mathrm{d}y$$

$$+\int_0^1 \left| \frac{-48\,T\cos(T(b-m_j-y(Z_j-m_j)))}{(b-m_j-y(Z_j-m_j))^4} \right.$$

$$\left. -\frac{48\sin(T(b-m_j-y(Z_j-m_j)))}{(b-m_j-y(Z_j-m_j))^5}+\frac{16T^2\sin(T(b-m_j-y(Z_j-m_j)))}{(b-m_j-y(Z_j-m_j))^3}\right|$$

$$\times I\{|b-m_j-y(Z_j-m_j)|\geqslant T^{-11/20}\}\,\mathrm{d}y$$

$$\leqslant 2T^{3.9}\frac{1}{|b-m_j||b-Z_j|}+48T^{3.2}+48T^{2.75}+16T^{3.65}.$$

Applying this inequality to (4.22), we obtain that

$$E\min\left(2w,\mathrm{e}\frac{|m_j-Z_j|}{2\pi(T^2+s^2)^2}\right.$$

$$\left.\times\int_0^1\left|\int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\right|\,\mathrm{d}y\right)$$

$$\leqslant 2wP\{|Z_j-m_j|<\ln(T)^{-1}\}+\ \mathrm{e}\frac{E|m_j-Z_j|}{2\pi(T^2+s^2)^2}$$

$$\times\left(2T^{3.9}\frac{1}{|b-m_j|\ln(T)}+\ 48T^{3.2}+48T^{2.75}+16T^{3.65}\right). \tag{4.23}$$

Therefore, it follows from (4.21), (4.23) and the conditions of Theorem 1

$$\int_{-T}^{T}\psi(s,b,\mathrm{i}t)(\varphi(s+\mathrm{i}t)-\varphi^*(s+\mathrm{i}t))\,\mathrm{d}t$$

$$\leqslant \mathrm{o}_T(1)+\mathrm{O}(1+U(b)), \tag{4.24}$$

where the constant in $\mathrm{O}(1+U(b))$ does not involve T and $\mathrm{o}_T(1)\to 0$ as $T\to\infty$.

One can show in a similar manner that

$$\int_{-T}^{T}\psi(s,b,\mathrm{i}t)(\varphi^*(s+\mathrm{i}t)-\varphi(s+\mathrm{i}t))\,\mathrm{d}t$$

$$=\sum_{j=1}^{\infty}g(j)E\min\Bigg(I\{m_j<b\}-I\{Z_j<b\}$$

$$+\mathrm{O}(\mathrm{e}^{-sZ_j}\min(1,(T(b-Z_j))^{-2}))$$

$$+\mathrm{O}(\mathrm{e}^{-sm_j}\min(1,(T(b-m_j))^{-2})),$$

$$\frac{Z_j - m_j}{2\pi(T^2+s^2)^2}\int_0^1 \mathrm{e}^{s(b-m_j-y(Z_j-m_j)}$$

$$\times \int_{-T}^{T}(T^2-t^2)^2\cos(t(b-m_j-y(Z_j-m_j)))\,\mathrm{d}t\,\mathrm{d}y\Bigg)$$

$$\leqslant \mathrm{o}_T(1)+\mathrm{O}(1+U(b)).$$

It is clear that this equality and (4.24) imply (4.16).

The proof of Lemma 4.3 is complete. □

5. Examples

Example 1. Let $x_1, x_2, \ldots,$ be a stationary AR(1) process, i.e.

$$x_i = \lambda x_{i-1} + \varepsilon_i, \quad x_0 \equiv 0, \ |\lambda| < 1,$$

where $\varepsilon_1, \varepsilon_2, \ldots,$ be a sequence of iid real-valued random variables with $E\varepsilon_i = 0$, $E\varepsilon_i^2 = 1$. A process of interest (Vexler and Dmitrienko, 1999) is

$$Z_j = \sum_{i=1}^{j} x_i^2.$$

Renewal-theoretic results for Z_j cannot be obtained by standard non-linear renewal theory (Lai and Siegmund, 1977, 1979; Woodroofe, 1990), because the difference between Z_j and the natural sequence of partial sums is not slowly changing. However, our Theorem 1 can be applied to obtain the following. Note that

$$Z_j = (1-\lambda^2)^{-1}\left(\sum_{k=1}^{j}\varepsilon_k^2 + 2\lambda\sum_{k=1}^{j}x_{k-1}\varepsilon_k - \lambda^2 x_j^2\right), \quad EZ_j \sim (1-\lambda^2)^{-1}j.$$

Define: $m_j = j/(1-\lambda^2)$. Thus: there exists a constant $c = c(\lambda, E\varepsilon_1^4)$ such that

$$U(b) = \frac{1}{b^2}\sum_{j=1}^{\infty} g(j)E\left(\sum_{i=1}^{j}x_i^2 - \frac{j}{1-\lambda^2}\right)^2 \mathrm{e}^{-j/(b(1-\lambda^2))}$$

$$\leqslant \frac{c}{b^2}\sum_{j=1}^{\infty} g(j)j\mathrm{e}^{-j/(b(1-\lambda^2))}.$$

For example, this expression is O(1), if $E\varepsilon_1^4 < \infty$ and $g(j)$ is bounded. For appropriate g, it is straightforward to verify that conditions (2.3)–(2.5) are satisfied.

Example 2. In many statistical inference problems, some predetermined accuracy is required of a procedure used, and the "optimal" fixed-sample-size procedure to meet this accuracy requirement often depends on some unknown nuisance parameter. In this case, the most frequently used sequential sampling scheme is the fully sequential sampling scheme due to Liu (1997), the Anscombe–Chow–Robbins scheme. In a particular case, the stopping time of the Anscombe–Chow–Robbins scheme may be written in the form

$$N(b) = \inf\left\{n \geqslant 1\colon \frac{n^\alpha L(n)}{S_n} \geqslant b\right\}, \qquad S_n = \mu + x_2 + \cdots + x_n,$$

where $\{x_i > 0,\ i > 1\}$ is an iid sequence with mean $\mu > 0$, $L(n)$ is a sequence of numbers given by $1 + L_0/n + o(1)$ as $n \to \infty$, $\alpha > 1$. The expectation of $N(b)$ has been studied by Woodroofe (1977) and, in particular, it has been shown that under certain assumptions

$$EN(b) = (\mu b)^{1/(\alpha-1)} + O(1), \qquad b \to \infty. \tag{5.1}$$

We are interested in an approximation to $EN(b)^2$ as $b \to \infty$. It is easy to see that

$$\begin{aligned}(EN(b))^2 \leqslant EN(b)^2 &\leqslant \sum_{j=1}^{\infty} j^2 P\{Z_{j-1} < b, Z_j \geqslant b\} + 1\\ &\leqslant \sum_{j=1}^{\infty}(2j+1)P\{Z_j < b\} + O(1),\end{aligned} \tag{5.2}$$

where $Z_j = j^\alpha L(j)/S_j$ and $b \to \infty$. Define: $m_j = j^{\alpha-1}L(j)/\mu$ and $Ex_2^8 < \infty$. Thus: there exists a constant $c = c(\mu, Ex_2^8)$ such that for all $j \geqslant 1$ and $0 < \epsilon < \mu$

$$\begin{aligned}E(Z_j - m_j)^2 &\leqslant L^2(j) j^{2(\alpha-1)}\left(\frac{E(\sum_{i=1}^{j}(x_i-\mu))^2}{\mu^2(\mu-\epsilon)^2 j^2}\right.\\ &\qquad\left.+\frac{E(\sum_{i=1}^{j}(x_i-\mu))^2 I\{S_j < (\mu-\epsilon)j\}}{\mu^4}\right)\\ &\leqslant c j^{2\alpha-3}.\end{aligned}$$

Therefore, for $g(j) = 2j+1$

$$\begin{aligned}U(b) &= \frac{1}{b^2}\sum_{j=1}^{\infty} g(j) E\left(Z_j - \frac{j^{\alpha-1}L(j)}{\mu}\right)^2 \exp\left(-\frac{j^{\alpha-1}L(j)}{b\mu}\right)\\ &\leqslant \frac{c}{b^2}\sum_{j=1}^{\infty}(2j+1) j^{2\alpha-3}\exp\left(-\frac{j^{\alpha-1}L(j)}{b\mu}\right) = O(b^{1/(\alpha-1)}).\end{aligned}$$

Applying Theorem 1, (5.1) and (5.2) we conclude

$$((\mu b)^{1/(\alpha-1)} + O(1))^2 \leqslant EN(b)^2 \leqslant \sum_{j=1}^{\infty}(2j+1)I\{m_j < b\} + O(b^{1/(\alpha-1)}),$$

$$EN(b)^2 = (b\mu)^{2/(\alpha-1)} + O(b^{1/(\alpha-1)}),$$

$$\mathrm{Var}(N(b)) = O(b^{1/(\alpha-1)}).$$

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